The Poncelet problem and the discrete-time pendulum

2009 J. Phys. A: Math. Theor. 42 175206

(http://iopscience.iop.org/1751-8121/42/17/175206)

IOP PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND THEORETICAL

J. Phys. A: Math. Theor. **42** (2009) 175206 (18pp) doi:10.1088/1751-8113/42/17/175206

The Poncelet problem and the discrete-time pendulum

Luc Vinet[1] **and Alexei Zhedanov**[2]

[1] Université de Montréal PO Box 6128, Station Centre-ville Montral QC H3C 3J7, Canada
[2] Donetsk Institute for Physics and Technology, Donetsk 83114, Ukraine

E-mail: luc.vinet@umontreal.ca and zhedanov@kinetic.ac.donetsk.ua

Received 13 November 2008, in final form 7 March 2009
Published 6 April 2009
Online at stacks.iop.org/JPhysA/42/175206

Abstract
We show that the Jacobi model for the famous Poncelet theorem in projective geometry is equivalent to Hirota's model of the discrete-time mathematical pendulum. We describe all types of solutions of Hirota's pendulum from a geometrical point of view. Apart from 'classical' pendulum solutions there are solutions with no continuous limit. These solutions have a very natural geometric treatment. The main result of the paper is that the generic fourth-order anharmonic oscillator admits direct time discretization preserving integrability. Equivalently, this means that there exists an integrable system on the unit circle which admits direct time discretization. This observation gives rise to a notion of the 'Maxwell caustic' corresponding to all systems of such type. Relations with spin chains and other problems in mathematical physics are considered.

PACS number: 02.30.Ik
Mathematics Subject Classification: 70H06

1. Jacobi model of the Poncelet theorem

The Poncelet theorem in elementary projective geometry is a statement about some recursive process on two arbitrary conics C and D in a projective plane [4]. Assume that conic C is located inside conic D. Choose an arbitrary point A_0 on D and pass the tangent line from A_0 to C (we choose one of two possible tangent lines). This tangent line intersects conic D in a new point A_1. Then, starting from A_1, we can continue this process passing a new tangent line from A_1 to conic C, obtaining a new point A_2 on D, etc. We then obtain a sequence $A_n, n = 0, 1, 2, \ldots,$ of points on conic D. The Poncelet theorem states that if the process is periodic, i.e. $A_N = A_0$ for some $N \geqslant 3$, then this property does not depend on the choice of the initial point A_0. In other words, the periodicity property (with the same N) will take place for *any* initial point A_0 on conic D.

Recently, it has been shown [7] that the Poncelet theorem plays an important role in the analysis of the Dirichlet problem for the string equation.

There are many different proofs of the Poncelet theorem (see, e.g., [4]). In our paper we will analyze only one of them belonging to Jacobi [17]. In what follows, we will use a 'modernized' version of the Jacobi approach contained in [6].

Jacobi used the property (already established by Poncelet himself) that the Poncelet theorem is projective invariant, so it is possible to choose an appropriate projective transformation in order to reduce the two conics to the simplest possible ones. There are different possibilities of how to do this. Jacobi chose the following one: both C and D are circles; D is a circle of radius R with the equation $x^2 + y^2 = R^2$, and C is a circle of a smaller radius $r < R$ located inside D with the equation $x^2 + (y - a)^2 = r^2$. Thus the centers of circles D and C have coordinates $(0, 0)$ and $(0, a)$, respectively. The parameter a is the distance between the centers of the two circles D and C.

We can parametrize points on circle D by an angle θ counted anti-clockwise from the 'bottom' point $(0, -R)$. Choose the starting point A_0 with the coordinate θ_0 and pass the tangent line (A_0, A_1) to circle C such that circle C will be to the left side with respect to the line (A_0, A_1). This means that the parameter θ_1 (corresponding to the point A_1) has restrictions: $0 < \theta_1 - \theta_0 < \pi$. We then repeat the process obtaining points with coordinates $\theta_2, \theta_3, \dots, \theta_n$. The Poncelet process will be periodic if $\theta_N = \theta_0$ for some $N \geqslant 4$.

Jacobi noticed the remarkable relation

$$R \cos\left(\frac{\theta_{n+1} - \theta_n}{2}\right) + a \cos\left(\frac{\theta_{n+1} + \theta_n}{2}\right) = r, \tag{1.1}$$

which is valid for all $n = 1, 2, \dots$. This relation follows easily from elementary geometric considerations. On the other hand, this relation can be considered as an 'integral' for the second-order nonlinear equation

$$(R - a) \tan\left(\frac{\theta_{n+1} + \theta_{n-1}}{4}\right) = (R + a) \tan\left(\frac{\theta_n}{2}\right), \qquad n = 2, 3, \dots \tag{1.2}$$

or in equivalent form

$$R \sin\left(\frac{\theta_{n+1} + \theta_{n-1} - 2\theta_n}{4}\right) = -a \sin\left(\frac{\theta_{n+1} + \theta_{n-1} + 2\theta_n}{4}\right). \tag{1.3}$$

Recall the notion of the 'integral' of the second-order nonlinear difference equation [10, 32, 33]. Let $f(x, y, z)$ be a function of three complex variables. We assume that this function possesses 'sufficiently good' analytical properties. Let x_n, $n = 0, \pm 1, \pm 2, \dots$, be a complex sequence. The equation,

$$f(x_n, x_{n-1}, x_{n+1}) = 0, \qquad n = 0, \pm 1, \pm 2, \dots, \tag{1.4}$$

can be considered as a second-order nonlinear difference equation with respect to the discrete variable x_n. We say that this equation possesses an integral, if there exists an analytic function $F(x, y)$ such that $F(x_n, x_{n+1}) = \text{const}$ for any given solution x_n of equation (1.4). Existence of the integral means that in the phase space $x = x_n$, $y = x_{n+1}$ all solutions of equation (1.4) belong to a family of curves $F(x, y) = C$, where the value of the constant C depends on the choice of initial conditions, say x_0, x_1. In general, for arbitrary algebraic functions $f(x, y, z)$, equation (1.4) is non-integrable (i.e., there is no analytical function $F(x, y)$ with the property $F(x_n, x_{n+1}) = \text{const}$). Existence of an integral $F(x, y)$ is the exception rather than a rule [32, 33].

Strictly speaking, equations (1.2) and (1.3) follow directly from relation (1.1) under additional restriction:

$$\sin\left(\frac{\theta_{n+1} - \theta_{n-1}}{4}\right) \neq 0. \tag{1.5}$$

But it is easily seen that in Jacobi's geometric construction (if $r > 0$) of the points A_n on circle D there are obvious restrictions $\theta_{n+1} > \theta_n$ and $\theta_{n+1} - \theta_{n-1} < 2\pi$. Indeed, the condition $\theta_{n+1} = \theta_{n-1}$ can be valid only if $r = 0$. But the case $r = 0$ is degenerated: it corresponds to solutions consisting only of two points on circle D: $\theta_{2n+1} = \theta_1, \theta_{2n} = \theta_2$. Thus for the Jacobi model condition (1.5) always holds.

Now we can present an explicit solution for the Poncelet process in the form (in [6], the solution is presented in an equivalent form through the Jacobi 'amplitude' function, $2\theta_n = \mathrm{am}(qn + \phi)$)

$$\cos(\theta_n/2) = cn(qn + \phi; k), \qquad \sin(\theta_n/2) = sn(qn + \phi; k), \tag{1.6}$$

where $cn(z; k)$ and $sn(z; k)$ are standard Jacobi elliptic functions (we use definition of [34]). It is elementary verified [6] that the parameters of elliptic functions are

$$k^2 = \frac{4aR}{(R+a)^2 - r^2}, \qquad cn(q; k) = \frac{r}{R+a}, \qquad dn(q; k) = \frac{R-a}{R+a}. \tag{1.7}$$

The parameter ϕ remains free; it describes the location of the initial point θ_1 on circle D.

Jacobi also noticed an important property of this Poncelet model. Construct a *linear pencil* $D + \lambda C$ of circles depending on a parameter λ. More exactly, this means that we consider a family of circles $C(\lambda)$ describing by the equation

$$x^2 + y^2 - R^2 + \lambda(x^2 + (y-a)^2 - r^2) = 0. \tag{1.8}$$

For $\lambda = 0$, we have circle D, for $\lambda = \infty$ one obtains circle C. For any nonzero positive value of the parameter λ we have a circle $C(\lambda)$ which lies between the circles C and D. One can say that the family $C(\lambda), \lambda > 0$ provides a *linear interpolation* between the circles D and C. An equation for circle $C(\lambda)$ can be presented in the form

$$x^2 + (y - a(\lambda))^2 = r^2(\lambda), \tag{1.9}$$

where

$$a(\lambda) = \frac{a\lambda}{\lambda+1}, \qquad r^2(\lambda) = \frac{(\lambda+1)(R^2 + \lambda r^2) - \lambda a^2}{(\lambda+1)^2}. \tag{1.10}$$

Consider now the Poncelet process for the circles D and $C(\lambda)$. From formulae (1.7) and (1.10) it is seen that $k^2(\lambda) = k^2$, i.e. the elliptic modulus does not depend on the parameter λ. The parameter $q(\lambda)$ will depend on λ:

$$dn(q(\lambda); k) = \frac{(R-a)\lambda + R}{(R+a)\lambda + R}.$$

We see, that $q(\lambda) \to 0$ when $\lambda \to 0$, i.e. the 'step' q of the linear grid $qn + \phi$ becomes small for small λ. Geometrically, this is obvious: when $\lambda \to 0$ circle $C(\lambda)$ becomes very close to circle D. Hence all the next points $A_1, A_2, \ldots$ of the Poncelet process will be close to the initial point A_0.

Thus, changing the parameter λ in the linear pencil $C(\lambda)$ we can vary the step q of the Poncelet process while the elliptic modulus k will be an invariant for the whole family $C(\lambda)$.

Find the periodicity condition for solution (1.6). Elliptic functions $sn(x; k)$ and $cn(x; k)$ have real half-period $2K(k)$, where $K(k)$ is the complete elliptic integral of the first kind: $sn(x + 2K; k) = -sn(x; k), cn(x + 2K; k) = -cn(x; k)$. Assuming that $qN = 2K(k)M$ for some positive integers $N, M < N$ we have $\cos(\theta_N/2) = (-1)^M \cos(\theta_0/2), \sin(\theta_N/2) = (-1)^M \sin(\theta_0/2)$. This means that $\theta_N = \theta_0 + 2\pi j$ with some integer j, whence angles θ_N and θ_0 describe the same position of the point on the circle and $A_N = A_0$. Thus the periodicity condition is

$$qN = 2K(k)M. \tag{1.11}$$

It is seen that the periodicity condition does not depend on ϕ; hence it does not depend on the choice of the initial point A_0.

2. The Jacobi model and Hirota's discrete-time pendulum

In 1982 R Hirota proposed [13] his model of a simple pendulum with discrete time. This model is remarkable, because it preserves all essential features of the ordinary (continuous time) model of a simple pendulum. We will follow the paper [21] where this Hirota model is described.

Recall that the ordinary simple pendulum is described by the Newton equation

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} = -\omega^2 \sin\theta, \tag{2.1}$$

where θ is the angle of pendulum with respect to the vertical line ($\theta = 0$ is the stability position of the pendulum), $\omega = (g/l)^{1/2}$, where l is the length of the mathematical pendulum, and g is the strength of the gravity field. The parameter ω is the angular frequency of small oscillations of the pendulum, i.e. for sufficiently small values of θ one has the solution

$$\theta(t) = C\cos(\omega t + \phi), \tag{2.2}$$

with C being an amplitude and ϕ a phase of the solution.

The total mechanical energy of the system,

$$E = l^2\,\dot\theta^2/2 + lg(1 - \cos\theta), \tag{2.3}$$

is an integral of the motion.

Harmonic solution (2.2) is valid only for small values of energy $E \ll lg$. For arbitrary values of E one has solutions in terms of elliptic functions.

If $E < 2lg$, then the solution is oscillator-like:

$$\sin(\theta/2) = k\, sn(\omega t + \phi; k), \qquad \cos(\theta/2) = dn(\omega t + \phi; k), \tag{2.4}$$

where $\omega^2 = g/l, k^2 = E/(2lg)$.

If $E > 2lg$, then we have the 'rotating' solution:

$$\cos(\theta/2) = cn(\omega t + \phi; k), \qquad \sin(\theta/2) = sn(\omega t + \phi; k), \tag{2.5}$$

where $k^2 = 2lg/E, \omega^2 = E/(2l^2)$.

If $E = 2lg$, then the solution will be

$$\sin(\theta/2) = 1/\cosh(\omega t + \phi), \qquad \cos(\theta/2) = \tanh(\omega t + \phi), \tag{2.6}$$

where $\omega^2 = g/l$.

The parameter ϕ is arbitrary. It corresponds to the initial conditions of the pendulum.

In Hirota's model, equation of motion (2.1) is replaced by its discretized version:

$$\sin\left(\frac{\theta(t+2\delta) + \theta(t-2\delta) - 2\theta(t)}{4}\right) = -\omega^2\delta^2 \sin\left(\frac{\theta(t+2\delta) + \theta(t-2\delta) + 2\theta(t)}{4}\right), \tag{2.7}$$

where δ is a parameter which defines a step of time discretization. When $\delta \to 0$ equation (2.7) returns to (2.1). Hirota showed [13, 21] that the discrete-time model (2.7) preserves all features of the simple pendulum: it admits an 'energy' integral

$$E = \frac{2}{\delta^2}\sin^2\left(\frac{\theta(t+\delta) - \theta(t-\delta)}{4}\right) + \omega^2\left(1 - \cos\left(\frac{\theta(t+\delta) + \theta(t-\delta)}{2}\right)\right), \tag{2.8}$$

and there exist solutions of all three types corresponding to the solutions of the simple pendulum for different values of the energy E.

Now we identify the Hirota pendulum with the Jacobi model. For this goal we need only to denote

$$\theta(t + 2\delta n) = \theta_n$$

for all $n = 0, 1, \ldots$. The Hirota equation (2.7) becomes the Jacobi equation (1.3) with $a/R = \omega^2\delta^2$. The 'energy' integral (2.7) coincides with the Jacobi integral with (say) $R = 1/\delta^2, a = \omega^2, E = R + a - r$.

Now we can consider a geometrical meaning of all possible solutions of Hirota's-Jacobi model.

The case $r < R, a < R - r$ corresponds to the Jacobi model when the inner circle with radius r is contained completely in the outer circle with radius R. In this case we have solution (1.6) corresponding to the 'rotating' solution (2.5) of the simple pendulum. Note that if $a \to 0$ then $k \to 0$ (as seen from (1.7)) and we have simple harmonic rotation $\theta_n = nq + \phi$ with $\cos q = r/R$. The physical meaning of this motion is obvious: remember that $a = \omega^2$. On the other hand $\omega^2 = g/l$. Thus $a \to 0$ means $g \to 0$, i.e. the case when gravity is neglectible, and the motion in this case is a free rotation with constant angular velocity. On the other side, the case $a = 0$ corresponds to two concentric circles. The discrete trajectory $\theta_0, \theta_1, \ldots$, in the Poncelet model is a sequence of rotations to the same angle. In the periodic case, this means that we have a regular N-gon inscribed in circle D.

Now consider the process when the inner circle C moves toward the boundary of circle D. The modulus k will tend to 1. In the limiting case $a = R - r$ we have that the inner circle has a tangent point $\theta = \pi$ with outer circle D. In this case, solution (1.6) is degenerated to

$$\cos(\theta_n/2) = 1/\cosh(qn + \phi), \qquad \sin(\theta_n/2) = \tanh(qn + \phi), \tag{2.9}$$

where $\cosh q = (2R - r)/r$. This solutions corresponds to solution (2.6) of the simple pendulum. From the geometrical point of view, this solution corresponds to a sequence of angles θ_n which have limiting point $\theta_\infty = \pi$ ('highest' position of the pendulum).

If we continue to move circle C upward, such that $a > R - r$ but $a < R + r$, then circle D will have two symmetric intersection points with circle C with angle coordinates $-\psi, \psi$, where

$$\cos\psi = \frac{r^2 - a^2 - R^2}{2aR}. \tag{2.10}$$

From formula (1.7) it follows that in this case $k > 1$. Using standard transformation formulae for elliptic functions (see, e.g., [34]) we can reduce this case to the 'usual' interval $0 < k < 1$ obtaining then the solution

$$\sin(\theta_n/2) = k\, sn(qn + \phi; k), \qquad \cos(\theta_n/2) = dn(qn + \phi; k), \tag{2.11}$$

where

$$k^2 = \frac{(R + a)^2 - r^2}{4aR}, \qquad dn(q; k) = r/(R + a). \tag{2.12}$$

This solution of the Jacobi model (or Hirota pendulum) corresponds to the oscillating solution (2.4) of the simple pendulum. Note that the solution is restricted in the region $-\psi \leqslant \theta \leqslant \psi$ as obvious from the geometrical point of view.

Consider now the case of small (harmonic) oscillations from the geometrical point of view. From formula (2.10) it is seen that $\psi > \pi/2$ if $r < R$. This means that the amplitude of the oscillations will be sufficiently large (more than $\pi/2$) and we will not achieve a harmonic regime in this situation. Hence, we need to consider the case $r > R$. As is easily seen from geometric considerations it is sufficient to choose arbitrary radius r such that $r > R$ and the parameter a which should be $a = r - R + \epsilon$, where $\epsilon \ll r - R$. Then admissible zone on circle D is located near the bottom point $\theta = 0$ and the amplitude $\psi^2 = 2\epsilon \frac{r}{R(r-R)}$ (up to terms of order of ϵ^2). In this situation, we can replace $\sin(\theta_n)$ or $\tan(\theta_n)$ with θ_n. Then equation (1.2) becomes

$$\theta_{n+1} + \theta_{n-1} = 2(2R - r)/r\, \theta_n. \tag{2.13}$$

This is a linear second-degree difference equation with general solution

$$\theta_n = C\ \cos(qn + \phi), \tag{2.14}$$

where

$$\cos(q) = (2R - r)/r.$$

This result can also be obtained directly from the 'oscillating' solution (2.11) because in this case $k \to 0$ as is seen from (2.12).

Note that the Jacobi model has one more type of solutions which has no analogs in the continuum limit. This type corresponds to the case when circle C lies completely outside circle D. This means that the condition $a > R + r$ holds.

In this case, we have solutions

$$\cos(\theta_n/2) = cn(qn + \phi), \qquad \sin(\theta_n/2) = (-1)^n sn(qn + \phi), \tag{2.15}$$

where modulus k is given by the same expression (1.7) (note that $k^2 < 1$ for $a > R$) and

$$dn(q; k) = \frac{a - R}{a + R}.$$

There are two limiting cases of this solution. In the first case, we have $a = R + r$; this means that two circles touch one another. The modulus becomes $k = 1$ and the solution is

$$\cos(\theta_n/2) = 1/\cosh(qn + \phi), \qquad \sin(\theta_n/2) = (-1)^n \tanh(qn + \phi)$$

with $\cosh(q) = (2R + r)/r$. Another limiting case corresponds to the inequality $a \gg R, r$. This means that circle C is 'removed to infinity'. Then the modulus $k^2 \ll 1$ and hence the solution becomes trigonometrical:

$$\cos(\theta_n/2) = \cos(qn + \phi), \qquad \sin(\theta_n/2) = (-1)^n \sin(qn + \phi) \tag{2.16}$$

with $\cos(q) = r/a \ll 1$. This means that q is a very small parameter and the solution is oscillating very slowly near the initial position.

If $a \gg R$ but $r/a = p < 1$ is finite, then we have $k = 0$, i.e. again trigonometric solution (2.16) but now $\cos(q) = p$, i.e. the parameter q is a finite. In another limiting case $a \gg r$ but $R/a = p$, we have $k^2 = 4p/(p+1)^2$. Thus $k^2 \leqslant 1$ and $k^2 = 1$ only if $a = R$. For $a > R$ (circle C lies outside circle D) we have $k^2 < 1$ and the solution will be presented by elliptic functions (2.15) but with $cn(q; k)$, i.e. $q = K(k)$. This solution is fully degenerated: $\theta_{2n} = \theta_0$ and $\theta_{2n+1} = \theta_1$ for $n = 0, 1, \ldots$. Geometrically, this is obvious: one can imagine $r = 0$ whereas R, a are finite ($a > R$). Then starting from any point on circle D, we can pass only one 'tangent' line to the infinitely small circle C. Thus the Poncelet process is degenerated: at any step we pass through circle C the only tangent line intersecting circle D in the same two points.

Finally, consider the limiting process when circle $C(\lambda)$ from the linear pencil will be close to circle D. This means that $\lambda \to 0$. As was explained in the previous section under such a process the elliptic modulus k becomes unchanged, while the parameter q tends to zero. Hence corresponding discrete trajectory $A_0, A_1, A_2, \ldots,$ will consist of infinitely small elements. In the limit $\lambda \to 0$, we thus obtain a system with continuous time—the simple pendulum described by equation (2.1).

It is important to note that this process can be 'reverted' in the following sense. Start from the ordinary pendulum (2.1) and take any solution, e.g. the 'rotating' one (2.5). 'Discretize' the time t replacing it by a linear grid: $t_n = q(n - n_0), n = 0, 1, 2, 3, \ldots,$ with some arbitrary real parameters q, n_0. We then obtain the sequence θ_n of points on the unit circle. These points will be related by equation (1.1), hence such direct discretization for the simple pendulum appears to be integrable and leads, in fact, to the Poncelet problem. This means that under discretization with an arbitrary step q we will have the property that corresponding discrete trajectory at its every step will touch some fixed circle. Location of this circle depends on the total energy E and on the step q of discretization. But for the fixed energy all circles corresponding to different values of the discretization parameter q lie in the linear Jacobi pencil (1.8). This remarkable property of the pendulum motion was first observed by J C Maxwell in a short note [19]. (Maxwell established this property in a slightly different but equivalent form) Therefore, it is reasonable to call this property the Maxwell property of discrete trajectories. In fact, Maxwell was the first who discovered nontrivial discretization of a one-dimensional mechanical system preserving the integrability property.

3. The Bertrand model and the classical XY-spin chain

In this section, we consider the Bertrand model of the Poncelet theorem and show how this model is related to the classical Heisenberg *XY*-spin chain.

It is well known that the Poncelet theorem is projective invariant. This means that the main statement of the Poncelet theorem will be true for any pair of conics obtained from the initial pair of conics (with the closure property) by an arbitrary projective transformation.

It is also well known that two arbitrary non-intersected (in a real projective space) conics C, D can be transformed into one of the two simplest cases:

(i) D is the unit circle and C is an arbitrary circle inside D; this is the Jacobi model;

(ii) conic D is again the unit circle and conic C is an ellipse *concentric* with D.

This model was considered by Bertrand [5] who obtained an alternative proof of the Poncelet theorem (for modern treatment of the Bertrand approach see, e.g., [23, 26]).

We describe briefly the Bertrand model (for details, see [23, 26]).

We can take the equation of ellipse C in the form

$$x^2/\alpha^2 + y^2/\beta^2 = 1, \tag{3.1}$$

where $0 < \beta < \alpha < 1$ are semi-axis of ellipse C. Start from the point A_0 with the coordinate θ_0 on the unit circle ($\theta = 0$ corresponds to the point $x = 0$, $y = -1$, as in the Jacobi model, with anti-clockwise agreement). Pass a straight tangent line from A_0 to ellipse C, get a new point θ_1 of intersection of this line with the unit circle D and then continue the Poncelet process obtaining points $\theta_2, \theta_3, \ldots$.

From elementary analytic geometry one can obtain the relation between θ_n and θ_{n+1}:

$$\cos(\theta_{n+1} - \theta_n) + (\alpha^2 - \beta^2)\cos(\theta_{n+1} + \theta_n) = \alpha^2 + \beta^2 - 1. \tag{3.2}$$

Relation (3.2) can be treated as an 'integral' corresponding to the Bertrand model. Shifting $n \to n - 1$ we obtain a nonlinear 'equation of motion':

$$\tan\left(\frac{\theta_{n+1} + \theta_{n-1}}{2}\right) = \frac{1 + \alpha^2 - \beta^2}{1 + \beta^2 - \alpha^2}\,\tan(\theta_n). \tag{3.3}$$

Strictly speaking, equation (3.3) is equivalent to relation (3.2) only under the condition

$$\sin((\theta_{n+1} - \theta_{n-1})/2) \neq 0. \tag{3.4}$$

But condition (3.4) obviously holds for the Bertrand model if $\alpha > 0$, $\beta > 0$.

We see that formally relation (3.2) is obtained from relation (1.1) for the Jacobi model under the substitution

$$\theta_n \to 2\theta_n, a/R = \alpha^2 - \beta^2, \qquad r/R = \alpha^2 + \beta^2 - 1.$$

Hence we can present the solution

$$\cos(\theta_n) = cn(qn + \phi; k), \qquad \sin(\theta_n) = sn(qn + \phi; k), \tag{3.5}$$

where

$$dn(q;k) = \frac{1+\beta^2-\alpha^2}{1+\alpha^2-\beta^2}, \qquad cn(q;k) = \frac{\alpha^2+\beta^2-1}{1+\alpha^2-\beta^2}, \qquad k^2 = \frac{\alpha^2-\beta^2}{\alpha^2(1-\beta^2)}. \tag{3.6}$$

Solution (3.5) corresponds to the case when ellipse D is located inside the unit circle C, i.e. $0 < \beta < \alpha < 1$.

There are two degenerated cases of this solution. If $\beta = \alpha$ then $k = 0$ and solution (3.5) becomes trigonometric:

$$\theta_n = qn + \phi, \qquad \cos q = 2a^2 - 1. \tag{3.7}$$

The geometrical meaning of this solution is obvious: if $\beta = \alpha$ then the inner ellipse becomes a concentric circle with radius $a < 1$.

Another limiting case appears when $\alpha \to 1$. In this case $k \to 1$ as seen from (3.6). Corresponding solution (3.5) becomes hyperbolic:

$$\cos(\theta_n) = 1/\cosh(qn+\phi), \qquad \sin(\theta_n) = \tanh(qn+\phi), \qquad \cosh(q) = \frac{2-b^2}{b^2}. \tag{3.8}$$

The geometrical meaning of this solution is also obvious: condition $a = 1$ means that the inner ellipse touches the unit circle in two points $\psi_{1,2} = \pm\pi/2$. All possible motions of the Poncelet process in this case are restricted by either the upper or the bottom semicircle. Solution (3.8) corresponds to the bottom semicircle (solution for the upper semicircle is obtained from (3.8) by inversion $\theta_n \to \pi + \theta_n$). Points $\psi_{1,2}$ are limiting points of the Poncelet process, e.g., for $n \to \infty$ we have $\theta_n \to \pi/2$.

Of course, it is possible to consider other possibilities, when ellipse C is not located purely inside of the unit circle D. This means that $\alpha > 1$, i.e., there are four intersection real points for the two conics D and C. The case $\beta > 1$ and $\alpha > 1$ has no geometrical meaning because in these cases (the unit circle D is located inside of ellipse C) all tangent points are imaginary.

Thus we have only one more type of solutions corresponding to the choice of parameters, $0 < \beta < 1 < \alpha$. In this case, we have oscillating solutions

$$\cos(\theta_n) = dn(qn+\phi;k), \qquad \sin(\theta_n) = ksn(qn+\phi;k), \tag{3.9}$$

where

$$cn(q;k) = \frac{1+\beta^2-\alpha^2}{1+\alpha^2-\beta^2}, \qquad dn(q;k) = \frac{\alpha^2+\beta^2-1}{1+\alpha^2-\beta^2}, \qquad k^2 = \frac{\alpha^2(1-\beta^2)}{\alpha^2-\beta^2}. \tag{3.10}$$

Geometrically, the Poncelet process is restricted by one of two symmetric arcs,upper or bottom, boundary points of which are the intersection points of the two conics. The limiting angles $(\pm\psi)$ of these arcs are found as

$$\sin(\psi) = k = \alpha\sqrt{\frac{1-\beta^2}{\alpha^2-\beta^2}}.$$

The Bertrand model of the Poncelet process has an interesting application in the theory of integrable magnetic chains. Consider the so-called classical Heisenberg XY-chain which is a set of $N+1$ two-dimensional vectors— 'spins'—$\mathbf{R}_n$, $n = 0, 1, \ldots, N$, with components $\mathbf{R}_n = (X_n, Y_n)$. We assume that these vectors belong to a unit circle, i.e.

$$\mathbf{R}_n^2 = X_n^2 + Y_n^2 = 1. \tag{3.11}$$

Each vector $\mathbf{R}_n$ is assumed to 'interact' with its nearest neighbors, i.e. with $\mathbf{R}_{n\pm1}$. Energy of this interaction is described by the Heisenberg-type expression [10]:

$$E = -\sum_{n=0}^{N-1}\{J_1 X_n X_{n+1} + J_2 Y_n Y_{n+1}\}, \tag{3.12}$$

where J_1, J_2 are some fixed constants described the anisotropy of interaction. In particular, the case $J_1 > 0, J_2 > 0$ corresponds to the so-called ferromagnetic model, and the case $J_1 < 0, J_2 < 0$ corresponds to the anti-ferromagnetic model. The problem is in finding all static solutions providing a local extremum of the function E. Boundary conditions may be periodic, i.e. $\mathbf{R}_N = \mathbf{R}_0$, free (i.e. no restrictions are imposed) and others (for example, one can fix initial $\mathbf{R}_0$ and final $\mathbf{R}_N$ vectors). Moreover, N may be infinity; in this case we deal with the unbounded Heisenberg chain.

Local extremes of function (3.12) are found by a standard procedure (e.g., by the Lagrange method of constrained extremum), and the corresponding equations (static Landau–Lifshitz equations) are [10]

$$J_1(X_{n-1} + X_{n+1}) = \lambda_n X_n, \qquad J_2(Y_{n-1} + Y_{n+1}) = \lambda_n Y_n, \tag{3.13}$$

where λ_n is the Lagrange multiplier needed to support the constraint (3.11).

It can easily be shown [10] that either $\lambda_n = 0$ for some n (in this case $\mathbf{R}_{n+1} = -\mathbf{R}_{n-1}$) or $\lambda_n \neq 0$ for all n. In the second case there is an 'integral', i.e. the following expression,

$$W = J_1^{-1} X_n X_{n+1} + J_2^{-1} Y_n Y_{n+1}, \tag{3.14}$$

does not depend on n on the solutions of the static equations (3.13).

There are two trivial 'homogeneous' solutions: $X_n \equiv 0$ and $Y_n \equiv 0$. These solutions are isolated from all nontrivial solutions of equations (3.13).

In what follows, we will consider only these, 'regular' solutions. Conversely, if expression (3.14) does not depend on n, we derive static equations (3.13) with $\lambda_n \neq 0$. Thus, for the regular case it is sufficient to restrict ourselves with studying integral (3.14) for all possible values of the parameter W.

The crucial observation is that integral (3.14) in the XY-spin chain coincides with integral (3.2) for the Bertrand model, if we identify vectors $\mathbf{R}_n$ with vertices of the Poncelet process on the conic (unit circle) D: $X_n = \sin(\theta_n)$, $Y_n = \cos(\theta_n)$. Correspondence of the parameters is

$$J_1^{-1} = \kappa(1 + \beta^2 - \alpha^2), \qquad J_2^{-1} = \kappa(1 + \alpha^2 - \beta^2), \qquad W = \kappa(\beta^2 + \alpha^2 - 1), \tag{3.15}$$

where κ is an arbitrary nonzero parameter.

Thus all regular solutions of the XY-spin chain are equivalent to solutions of the Bertrand model. All solutions of the Bertrand model provide a nontrivial geometric interpretation of the solutions of the XY-chain.

In particular, all regular solutions of the XY-chain have the property: there is an ellipse, concentric with the unit circle such that the 'trajectory' (i.e. sequence of straight lines defined by the vectors $\mathbf{R}_{n+1} - \mathbf{R}_n$, $n = 0, 1, \ldots$) touches this ellipse in its every branch. This property is not obvious *a priori* from solutions of the XY-chain.

We can assume that $\kappa = 1$, $0 < J_2 < J_1$. This choice corresponds to the ferromagnetic chain with equilibrium (i.e. with minimal energy) solution $Y_n = -1$, $X_n = 0$ for all n (note that this solution is twice degenerated: $Y_n = 1$, $X_n = 0$ is also the minimal solution).

If $J_1 = J_2$, (i.e. the spin chain becomes isotropic) we have $\alpha = \beta$, i.e. the inner ellipse becomes a circle and we obtain trigonometric solutions (3.7). This corresponds to the well-known fact that for the isotropic spin chain all regular solutions have the form (3.7).

Solution (3.5) of the Bertrand model corresponds to the case $|W| < J_1^{-1}$ and oscillating solution (3.9) corresponds to the case $J_1^{-1} < W < J_2^{-1}$.

If $W = J_2^{-1}$ we obtain the solution describing the domain wall [10]. It corresponds to solution (3.8) of the Bertrand model.

The case of closed (periodic) spin chain $\mathbf{R}_N = \mathbf{R}_0$ corresponds to the Poncelet N-gon in the Bertrand model. The Poncelet theorem in this case states that there are infinitely many regular solutions of the closed XY-chain corresponding to a fixed value of the integral W. More exactly, this means the following. Assume that there exists a solution of the periodic spin chain with N spins corresponding to some fixed value W. We can characterize this solution by the direction of the first spin $\mathbf{S}_0$. Then there are infinitely many solutions of the same periodic spin chain with the same value W. All this solutions correspond to an arbitrary position of the initial vector $\mathbf{S}_0$. Indeed, by the Poncelet theorem, in the Bertrand model all initial positions of the point on the unit circle are admissible if the closed N-gon exists. Note that for fixed values of the interaction parameters J_1, J_2 and fixed number N there are finitely many values of the parameter W for which such a solution can exist. Indeed, according to the determinant Cayley criterion (see, e.g., [7]), in the Bertrand model, for the given N there is a polynomial equation for admissible values of the parameters α, β necessary and sufficient for the Poncelet N-gon to exist. On the other hand, in the XY-model there are two independent parameters, say J_2/J_1 and W (clearly, we can put $\kappa = 1$ in (3.15) without loss of generality) which are equivalent to two independent parameters α, β in the Bertrand model. Hence, there is a polynomial equation relating parameters J_2/J_1 and W. Usually the ratio J_2/J_1 is fixed by physical conditions (the parameters J_1, J_2 depend on the intrinsic properties of the XY-model). Then the admissible values of the parameter W can be found from solving a polynomial equation with coefficients depending on J_2/J_1.

We thus see that the Poncelet theorem about N-gons has a very natural interpretation in terms of periodic solutions of the spin XY-chain.

Note that in the continuous limit the Bertrand model (or XY-model) is reduced to the Neumann model on the circle [32] which describes a point on the circle moving under the quadratic (oscillator) potential on the plane $U(x, y) = \xi_1 x^2 + \xi_2 y^2$ with generic constants ξ_1, ξ_2. Equivalently, the Neumann system on the circle is described by the energy equation

$$\dot{\theta}^2 = \alpha_0 + \alpha_1 \cos(2\theta), \tag{3.16}$$

which can formally be obtained from the corresponding equation for the pendulum by a substitution $\theta \to 2\theta$. Note that in the Greenhill monograph such a system is called 'quadrantal oscillations' [11]. Of course, explicit solutions of the Neumann system are given by the Jacobi elliptic functions (see [11] for details).

4. Projective transformations of the unit circle

In what follows we will assume that $R = 1$, i.e. circle D is the unit circle. The Bertrand and the Jacobi models of the Poncelet problem are then connected by a projective transformation of the two-dimensional plane which preserves the unit circle $x^2 + y^2 = 1$. In this section, we describe briefly the group of projective transformations preserving the unit circle. Of course, this subject is well known; see, e.g., [3]; we would like to stress a striking analogy between this group and the Lorentz group $O(2, 1)$ which makes derivation of all needed formulae from projective geometry transparent from the 'relativistic' point of view.

Indeed, the unit circle can be described as a projective curve:

$$x_1^2 + x_2^2 - x_0^2 = 0, \tag{4.1}$$

where we introduced the projective coordinates $x = x_1/x_0$, $y = x_2/x_0$. We can interpret equation (4.1) as an equation of the 'light cone' in the Minkowski space with coordinates x_0, x_1, x_2. The projective transformations preserving the unit circle are linear transformations of the coordinates x_0, x_1, x_2 preserving equation (4.1). Equivalently, these transformations can be interpreted as the Lorentz transformations preserving the light cone. Up to inessential scaling transformation $x_i \to \gamma x_i$, $i = 0, 1, 2$, these Lorentz transformations can be described as the group $O(2, 1)$ containing three elementary one-parametric subgroups:

(i) simple rotations:

$$x_0 \to x_0, \qquad x_1 \to x_1 \cos(\phi) + x_2 \sin(\phi), \qquad x_2 \to -x_1 \sin(\phi) + x_2 \cos(\phi),$$

(ii) 'boost' transformation leaving the coordinate x_2 unchanged:

$$x_0 \to x_0 \cosh(\chi) + x_1 \sinh(\chi), \qquad x_1 \to x_1 \cosh(\chi) + x_0 \sinh(\chi), \qquad x_2 \to x_2,$$

(iii) 'boost' transformation leaving the coordinate x_1 unchanged:

$$x_0 \to x_0 \cosh(\nu) + x_2 \sinh(\nu), \qquad x_x \to x_2 \cosh(\nu) + x_0 \sinh(\nu), \qquad x_1 \to x_1.$$

Returning to the Euclidean coordinated x, y we see that the transformation (i) corresponds to a rotation of the plane by the angle θ:

$$x \to x \cos(\phi) + y \sin(\phi), \qquad y \to -x \sin(\phi) + y \cos(\phi).$$

'Boost' (ii) corresponds to the nontrivial transformation of the plane:

$$x \to \frac{x \cosh(\chi) + \sinh(\chi)}{x \sinh(\chi) + \cosh(\chi)}, \qquad y \to \frac{y}{x \sinh(\chi) + \cosh(\chi)},$$

while boost (iii) corresponds to the projective transformation:

$$x \to \frac{x}{y \sinh(\nu) + \cosh(\nu)}, \qquad y \to \frac{y \cosh(\nu) + \sinh(\nu)}{y \sinh(\nu) + \cosh(\nu)}.$$

Note that boost (iii) (under an appropriate choice of the parameter ν) allows one to transform the Jacobi model into the Bertrand model. Equivalently, this means that one can find a transformation of the class (iii) which transforms an arbitrary circle inside the unit circle D (with the center located at the axis OY) into the ellipse concentric with D (see [23] for details).

Consider also the transformation law for points on the unit circle.

It is convenient to use the standard rational parametrization of the unit circle:

$$x = \frac{1-u^2}{1+u^2}, \qquad y = \frac{2u}{1+u^2}, \tag{4.2}$$

where $u = \tan(\theta/2)$ (in this case the angle $\theta = 0$ corresponds to the point $x = 1, y = 0$ on the unit circle).

Then it is elementary verified that under the rotation (i) the parameter u is transformed as

$$\tilde{u} = \frac{u+\sigma}{1-\sigma u},$$

where $\sigma = \tan(\phi/2)$.

Under boost (ii) we have simple scaling transformation:

$$\tilde{u} = u\,\mathrm{e}^{-\chi}.$$

Finally, under boost (iii) we have the transformation

$$\tilde{u} = \frac{u+\tau}{1+u\tau},$$

where $\tau = \tanh(\nu/2)$. Now consider the general projective transformation (which is a combination of elementary transformations (i)—(iii)) preserving the unit circle. It is seen that under such a transformation we have general Möbius transformation,

$$\tilde{u} = \frac{au+b}{cu+d}, \tag{4.3}$$

of the parameter u with general real coefficients a, b, c, d.

Thus general projective transformations conserving the unit circle generate the general Möbius group (4.3) on this circle. This observation will be useful in the following section when we construct a generic discrete-time integrable system on the unit circle.

5. A discrete integrable system on the unit circle which generalizes a simple pendulum

A simple pendulum is described by the Newton equation (2.1) or, equivalently, by the energy (we put $l = 1$ which can also be achieved by an appropriate scaling transformation of the time t)

$$E = \dot{\theta}^2/2 + \omega^2(1 - \cos\theta). \tag{5.1}$$

Applying an arbitrary projective transformation preserving the unit circle, we can transform a simple pendulum to another dynamical system on the unit circle having explicit solutions in terms of elliptic functions.

Indeed, the energy equation (5.1) in terms of the parameter u can be written as

$$2\dot{u}^2 = E(1 + u^2)^2 - 2\omega^2 u^2(1 + u^2). \tag{5.2}$$

Under the Möbius transform (4.3) this equation becomes

$$\dot{u}^2 = P_4(u), \tag{5.3}$$

where $P_4(u)$ is a generic polynomial of degree $\leqslant 4$ in the argument u. Returning to the variable θ we obtain the equation

$$\dot{\theta}^2 = U(\theta) = \alpha_0 + \alpha_1 \sin(\theta) + \alpha_2 \cos(\theta) + \beta_1 \sin(2\theta) + \beta_2 \cos(2\theta), \tag{5.4}$$

with arbitrary constants $\alpha_0, \alpha_1, \alpha_2, \beta_1, \beta_2$. We thus obtained a general integrable system on the unit circle with the potential $U(\theta)$ which can be integrated in terms of elliptic functions.

In order to clarify the physical meaning of the dynamical system obtained we can rewrite the potential function $U(\theta)$ in terms of the Cartesian coordinates $x = \sin\theta$, $y = -\cos\theta$. We then have

$$U(x, y) = b_1 x^2 + b_2 y^2 + b_3 xy + a_1 x + a_2 y + a_0 \tag{5.5}$$

with arbitrary constants $a_1, \ldots, b_2$ (these constants are easily related to $\alpha_0, \ldots, \beta_2$). The potential function (5.5) corresponds to motion in the generic potential with all possible terms up to second degree in the variables x, y restricted to the unit circle. The linear terms, $a_1 x + a_2 y$, correspond to the uniform gravity field of an arbitrary direction. The quadratic terms, $b_1 x^2 + b_2 y^2 + b_3 xy$, correspond to some anisotropic oscillator-like potential. If all quadratic terms vanish (i.e. if $b_1 = b_2 = b_3 = 0$) then we return to the ordinary pendulum (with rotated equilibrium state). If all linear terms are absent (i.e. $a_1 = a_2 = 0$) we obtain the so-called Neumann system on the unit circle [32]. Note that the ordinary Neumann system contains only diagonal terms $U(x, y) = b_1 x^2 + b_2 y^2$; it is clear, however, that the term $\propto xy$ can be eliminated from (5.5) by a rotation of the unit circle by an appropriate angle. We thus see that potential function (5.5) describes a superposition of the simple pendulum and the Neumann system on the unit circle.

Consider now the corresponding discrete-time model. Any projective transformation sends conics to conics and straight lines to straight lines. Moreover, any tangent line to a conic is transformed into a tangent line to a conic. Hence, the Maxwell property of the pendulum will be preserved under an arbitrary projective transformation preserving the unit circle.

This means the following. Assume that $\theta(t)$ is a solution corresponding to the dynamical system (5.4) with arbitrary initial conditions. Take an arbitrary real parameter $h > 0$ and consider a discrete set of points $\theta_n = \theta(t_0 + hn)$, $n = 0, 1, 2, \ldots,$ on the unit circle. Then any line passing through the points θ_n, θ_{n+1} will touch a fixed conic for all n. All these conics form a linear pencil $C(\lambda)$ which can be defined by the equation $D + \lambda C = 0$, where $D = 0$ is the equation of the unit circle and $C = 0$ is the equation of the corresponding fixed conic. The parameter λ depends on the discretization step h. When $h \to 0$ then $\lambda \to 0$ and conics $C(\lambda)$ tends to the unit circle D.

We can summarize this observation in

Proposition 1. *The dynamical system on the unit circle with the potential function (51) admits the same Maxwell property for 'discretized' trajectories as for the ordinary pendulum.*

What is a discrete dynamical system describing the obtained points θ_n? Again it is more convenient to deal with variables $u_n = \tan(\theta_n/2)$. Then the Jacobi integral (1.1) can be presented in the form

$$F(u_n, u_{n+1}) = (R + a + (R - a)u_n u_{n+1})^2 - r^2\left(1 + u_n^2\right)\left(1 + u_{n+1}^2\right) = 0, \quad (5.6)$$

where $F_J(x, y)$ is a special case of general symmetric biquadratic polynomial,

$$F(x, y) = \sum_{i,k=0}^{2} a_{ik} x^i y^k, \quad (5.7)$$

with the symmetric matrix $a_{ik} = a_{ki}$.

Performing the generic Möbius transformation for u_n, u_{n+1} we can obtain the generic biquadratic equation $F(u_n, u_{n+1}) = 0$ starting from the Jacobi equation (5.6). Thus the discrete dynamical system corresponding to time discretization of equation (5.4) on the unit circle is described by the generic biquadratic equation,

$$F(u_n, u_{n+1}) = \sum_{i,k=0}^{2} a_{ik} u_n^i u_{n+1}^k = 0. \quad (5.8)$$

In order to derive the corresponding second-order difference equation we consider the same biquadratic equation $F(u_n, u_{n-1}) = 0$ obtained from (5.8) by the shift $n \to n - 1$. Subtracting these equations and assuming the condition $u_{n-1} \neq u_{n+1}$, we obtain the equation

$$u_{n-1}u_{n+1}\Phi_2(u_n) + (u_{n-1} + u_{n+1})\Phi_1(u_n) + \Phi_0(u_n) = 0, \quad (5.9)$$

where $\Phi_i(x)$, $i = 0, 1, 2$, are quadratic polynomials easily related to the biquadratic function $F(x, y)$. Equation (5.9) is linear with respect to both variables u_{n+1} and u_{n-1}. This means that starting from the fixed initial points u_0, u_1 we can determine from (5.9) uniquely step by step all further points $u_2, u_3, \ldots$. Equation (5.9) is a generalization of equation (1.2) for the Jacobi model and (3.3) for the Bertrand model.

Discrete integrable systems of type (5.9) were intensively studied in many papers during last 25 years. These systems are connected with rational integrable maps of the two-dimensional plane to itself [16, 33].

Reciprocal statement is also interesting.

Proposition 2. *Start from the discrete dynamical system given by (5.8). Assume that this system has infinite families of real solutions $u_0, u_1, u_2, \ldots$. We also assume the condition $u_{n+1} \neq u_{n-1}$ for all $n = 0, 1, 2, \ldots$. Let us identify u_n with a point θ_n on the unit circle D by the standard substitution $u_n = \tan(\theta_n/2)$. Then all lines passing through neighbor points θ_n, θ_{n+1} will touch some conic C.*

This means that our discrete system on the unit circle is equivalent to the Poncelet problem for the unit circle D and conic C. Consider the pencil of conics $C(\lambda) = D + \lambda C$. Then for every $0 < \lambda < \infty$ we will obtain a family of similar discrete dynamical system of the type

$$F(u_n, u_{n+1}; \lambda) = \sum_{i,k=0}^{2} a_{ik}(\lambda) u_n^i u_{n+1}^k = 0, \quad (5.10)$$

where the coefficients $a_{ik}(\lambda)$ of the biquadratic function $F(x, y; \lambda)$ are some functions in λ. For $\lambda \to 0$ we obtain a dynamical system (5.4) with continuous time on the circle.

As a by-product of these consideration we can obtain a generalization of another result by Hirota concerning the discrete-time model of the anharmonic oscillator. Indeed, in the same work [13] Hirota obtained a remarkable result: if we take a one-dimensional classical anharmonic oscillator with the Hamiltonian

$$H = p^2/2 + \alpha x^4 + \beta x^2 + \gamma,$$

then the direct discretization of the motion $x_n = x(t_0 + nh)$ (with an arbitrary discretization step) leads to an integrable discrete system for variables x_n. More exactly, variables x_n, x_{n+1} are connected by a symmetric biquadratic relation $F(x_n, x_{n+1}) = 0$, where the function $F(x, y)$ is a special case of biquadratic containing only terms, $x^2y^2, x^2 + y^2, xy, 1$ (see [15, 21]) for details. The reason of such integrability follows directly from our results, because Hirota's anharmonic oscillator is a special case of a one-dimensional system (5.3), where the polynomial $P_4(x)$ contains only terms with even degree: $P_4(x) = a_4x^4 + a_2x^2 + a_0$. Hence Hirota's anharmonic oscillator admits mapping to the unit circle $x_n = \tan(\theta_n/2)$ leading to the already studied integrable system (with both continued and discrete time).

But Hirota's example admits generalization to arbitrary polynomial $P_4(x)$ of fourth degree. So we have

Proposition 3. *The generalized anharmonic oscillator of the type*

$$\dot{x}^2 = P_4(x),$$

where $P_4(x)$ is an arbitrary fourth-degree polynomial, admits direct time discretization leading to a discrete integrable system. Namely, let $x(t)$ be an arbitrary solution of this system. Introduce the discrete set of variables $x_n = x(t_0 + hn)$ with arbitrary parameters t_0, h. Then variables x_n satisfy the relation $F(x_n, x_{n+1}) = 0$, where $F(x, y)$ is a generic biquadratic curve (53). The coefficients a_{ik} of the polynomial $F(x, y)$ depend on the coefficients of the polynomial $P_4(x)$ and on the parameter h (but not on the parameter t_0).

We give here another—more direct—proof of this proposition. It is well known [34] that the equation $\dot{x}^2 = P_4(x)$ has a generic solution $x(t) = \phi(t)$ expressed in terms of the second-order elliptic function $\phi(t)$ (i.e. double-periodic meromorphic function having exactly two poles in the fundamental parallelogram of periods). The simplest examples of the second-order elliptic functions are the Weierstrass function $\phi(t) = \wp(t)$ which has one double pole at $t = 0$ and the Jacobi function $\phi(t) = sn(t)$ which has two simple poles at $t = \mathrm{i}K'$ and $t = 2K + \mathrm{i}K'$. Vice versa, any second-order elliptic function $\phi(t)$ satisfies the differential equation $\dot{\phi}^2 = P_4(\phi)$ with some polynomial $P_4(x)$ of degree 4 or 3. Assume that $\phi(t)$ is an arbitrary second-order elliptic function. Consider two functions, $x(t) = \phi(t)$ and $y(t) = \phi(t + h)$, where h is an arbitrary complex parameter. Then it is well known [34] that functions $x(t), y(t)$ satisfy the equation $F(x, y) = 0$, where $F(x, y)$ is a biquadratic polynomial. Thus for any time discretization step h we obtain the corresponding discrete integrable system in the phase space x_n, x_{n+1}, where $x_n = x(t + nh)$, $n = 0, 1, 2, \ldots$.

Note that the generic symmetric biquadratic equation (5.10) appears naturally in many branches of mathematics and mathematical physics. We mention, e.g., the Baxter approach to exactly solvable models in statistical physics where relation (5.10) is crucial in solving the eight-vertex model [2].

Relation (5.10) appears also in the following problem from the theory of partial differential equations. Assume that we have a closed domain U on the plane bounded by a curve Γ. The so-called John algorithm allows one to recognize when the Dirichlet problem for the string equation in some domain U will have only unique solution. So far, explicit solutions for the John algorithm were known only for a rectangle and an ellipse. In [7], it was shown that the John algorithm for the domain U bounded by the generic biquadratic curve is equivalent to the discrete-time dynamical system described by (5.10). Non-unique solvability in this case is equivalent to the existence of periodic N-gon solutions of the Poncelet problem (see [7] for further details).

In theory of biorthogonal rational functions (BRF) relation (5.10) is a crucial tool in generating the so-called elliptic grids which are important in construction of families of BRF with some good properties [28].

6. Possible generalizations and concluding remarks

In [27], Sogo proposed a discrete version of the Euler elastic problem. He introduced the chain of small solid rods located on the plane XY. The direction of the nth rod is given by the angle θ_n. The interaction energy of such a chain is chosen as

$$E = \sum_n \sin^2((\theta_{n+1} - \theta_n)/4) - \varepsilon \sin^2((\theta_{n+1} + \theta_n)/4), \tag{6.1}$$

where $0 < \varepsilon < 1$ is a parameter modeling the discrete elastic properties of the chain. Then the extremum conditions for this chain lead to the equilibrium equations

$$\tan((\theta_{n-1} + \theta_{n+1})/4) = \frac{1-\varepsilon}{1+\varepsilon} \tan(\theta_n/2). \tag{6.2}$$

This equation coincides with the Jacobi equation (1.2) where $\varepsilon = -a/R$. Under the above-mentioned restriction for ϵ this means that $-R < a < 0$. In terms of the corresponding discrete pendulum model this means that $\omega^2 < 0$ or, equivalently, $g < 0$ which corresponds to reverting the direction of the gravity force. Thus the Sogo discrete model of an elastic rod is essentially equivalent to the Jacobi–Poncelet model or the Hirota discrete pendulum model. This is not surprising because it is well known that the ordinary Euler elastic model is closely related to the ordinary pendulum [25].

Veselov [32] proposed a variational (Lagrangian) approach to the discrete-time integrable systems with the interaction energy of the form

$$E = \sum_n W(q_n, q_{n+1}), \tag{6.3}$$

where $W(x, y)$ is a function of two variables and q_n a dynamical variable parametrized by the discrete-time variable $n = 0, \pm 1, \pm 2, \ldots$. Static (extremal) equations are derived from (6.3): $\frac{\partial E}{\partial_{q_n}} = 0$ or, explicitly,

$$\frac{\partial W}{\partial x}(q_n, q_{n+1}) + \frac{\partial W}{\partial y}(q_{n-1}, q_n) = 0. \tag{6.4}$$

The system is called *integrable* if an additional relation

$$F(q_n, q_{n+1}) = \text{const} \tag{6.5}$$

holds for all $n = 0, \pm 1, \pm 2, \ldots,$ with some analytic function $F(x, y)$. All models we considered here are integrable in this sense. We already know that these models can be transformed into one another by projective transformations of the plane. Nevertheless, the Lagrangians E (6.3) corresponding to these models (say, the Jacobi and Bertrand models of the Poncelet problem) cannot be obtained from one another by a projective transformation. The reason is that the projective transformations do not preserve the length. Hence for every concrete integrable model one should find the corresponding Lagrangian separately.

Consider, e.g., that the Poncelet model can be realized in terms of two confocal ellipsis [32]. This model is equivalent to the so-called billiard model for the Poncelet problem [9, 30]. In this model, the two conics C and D are chosen as confocal ellipses. The boundary of ellipse D is considered as a perfect billiard. This means that a mechanical particle moves without interaction inside ellipse D (i.e. its trajectory is a piecewise straight line) and reflects by the mirror law from its boundary. Then it can easily be shown that such trajectory will be tangent to a second ellipse C which is confocal with ellipse D. This gives another simple mechanical model for the Poncelet theorem. For details see, e.g., [9, 30, 32]. (The authors are grateful to the referee for drawing their attention to the recent book [9]) In this model the Lagrangian coincides with the total length of the discrete trajectory [32]:

$$E = \sum_n |\mathbf{r}_n - \mathbf{r}_{n+1}|.$$

Indeed, it is well known that in this model the geometric length of corresponding Poncelet polygons possesses nice extremal properties [20]. On the other hand, the Lagrangian for the Bertrand model (or, equivalently, spin XY-chain) is given by expression (3.12) having no relation with the length of vectors. The problem of construction of the Lagrangian corresponding to the given integrable discrete-time model remains an interesting open problem.

Another open problem is the extension of the Maxwell property to three-dimensional potentials (either free or restricted by a sphere or a more general surface). For non-oscillator potentials in the free space, the time discretization procedure is a nontrivial problem. For example, for the Kepler problem one first needs to perform the so-called regularization procedure, passing to a new time depending on the coordinates of the moving particle (see [22] for details). A general discussion concerning the discretization procedure in classical mechanics can be found in [29].

We can define the *Maxwell caustic* as follows. Assume that a trajectory $\mathbf{r}(t)$ of a particle in some potential is a curve Γ. The curve Γ may be either as a result of a motion in some potential on the Euclidean plane (e.g. the Kepler ellipse in the Newtonian potential) or a prescribed constraint (e.g. pendulum motion on the unit circle). Perform time discretization with the step h, i.e. consider the discrete set of vectors $\mathbf{r}_n = \mathbf{r}(t_0 + hn)$, where t_0 (the initial value of time) is a fixed parameter. Then the Maxwell caustic $\mathcal{M}_h$ is a curve such that every straight line $\mathbf{r}_{n+1} - \mathbf{r}_n$ touches the curve $\mathcal{M}_h$. Of course, the Maxwell caustic depends on the discretization parameter h, and $\mathcal{M}_h \to \Gamma$ if $h \to 0$. We say that the motion possesses the *Maxwell property* if the set of curves $\mathcal{M}_h$ for all h can be described as a linear pencil, i.e. every Maxwell caustic can be presented in the form

$$\mathcal{M}_h = \Gamma + \lambda(h)\mathcal{M}_{h_0},$$

where h_0 is a fixed parameter and $\lambda(h)$ is a scalar continuous function of h such that $\lambda(0) = 0$. Clearly, if the Maxwell property holds then the basic parameter h_0 can be chosen arbitrarily from all admissible set h.

We already know that the Maxwell property holds for the simple pendulum and, more generally, for the dynamical system on the unit circle described by the potential (5.5). In all these cases the Maxwell property is related to the Poncelet problem, i.e. the set of the Maxwell caustics is a linear pencil of conics.

Note that for elastic billiards restricted by a closed curve $\mathcal{D}$ one can introduce caustics by the similar definition [9, 30]: a caustic $\mathcal{C}$ of a plane billiard is a curve such that if a billiard trajectory is tangent to it, then it remains tangent to it after every reflection. For elastic billiards it is assumed that trajectory is a set of straight lines satisfying the mirror law: the angle of incidence with $\mathcal{D}$ equals the angle of reflection. We say that the billiard caustics $\mathcal{C}$ satisfy the Maxwell property if they all belong to the same linear pencil. This pencil will include the billiard curve $\mathcal{D}$, of course. It is well known that all billiard caustics inside the elliptic billiard are confocal ellipses [30], and hence they satisfy the Maxwell property.

The open problem is to describe all two-dimensional (plane) motions having the Maxwell property. In particular, it would be interesting to classify all plane billiards with this property.

We can propose two conjectures.

Conjecture 1. The potential (5.5) is the most general one admitting the Maxwell property for the Maxwell caustics in the unit circle.

Conjecture 2. The elliptic billiards are the only ones admitting the Maxwell property for billiard caustics.

Another interesting problem is how to generalize the Maxwell property to three-dimensional motions. We mention an interesting approach to the explicit time discretization of the Euler top proposed by Hirota and Kimura [14] and developed further in [24].

Finally, note that pencils of conics appear naturally in theory of the Yang–Baxter maps and multi-dimensional integrable billiards [1, 8].

Acknowledgments

The authors thank to the referees for valuable remarks. AZh thanks the Centre de recherches mathématiques of the Université de Montréal for hospitality and V Burskii, Y Minesaki, Y Nakamura and S Trsujimoto for such a helpful discussion.

References

[1] Adler V E, Bobenko A I and Suris Yu B 2004 Geometry of Yang–Baxter maps: pencils of conics and quadrirational mappings *Commun. Anal. Geom.* **12** 967–1007

[2] Baxter R 1982 *Exactly Solvable Models in Statistical Mechanics* (London: Academic)

[3] Beardon A 1983 *The Geometry of Discrete Groups* (Graduate Texts in Mathematics) vol 91 (New York: Springer)

[4] Berger M 1978 *Géométrie* (Paris: CEDIC)

[5] Bertrand J 1870 *Traité de Calcul Différentiel et de Calcul Intégral* vol 2 (Paris: Gauthier-Villars)

[6] Bos H J M, Kers C, Oort F and Raven D W 1987 Poncelet's closure theorem *Expositiones Math.* **5** 289–364

[7] Burskii V and Zhedanov A 2006 On Dirichlet problem for string equation, Poncelet problem, Pell-abel equation, and some other related problems *Ukrainian Math. J.* **58** 487–504

[8] Dragovic V and Radnovic M 2006 Geometry of integrable billiards and pencils of quadrics *J. Math. Pures Appl.* **85** 758–90

[9] Flatto L 2009 *Poncelet's Theorem* (Providence, RI: American Mathematical Society)

[10] Granovskii Ya I and Zhedanov A S 1986 Integrability of the classical XY-chain *Pis'ma Zh. Eksp. Teor. Fiz.* **44** 237–9 (Russian)

Granovskii Ya I and Zhedanov A S 1986 Integrability of the classical XY-chain *JETP Lett.* **44** 304–7 (Engl. Transl.)

[11] Greenhill A G 1959 *The Applications of Elliptic Functions* (New York: Dover)

[12] Griffiths P and Harris J 1978 On a Cayley's explicit solution to Poncelet's porism *Enseign. Math.* **24** 31–40

[13] Hirota R 1982 *Tech. Rep. No.* A-12 Hiroshima Univ.

[14] Hirota R and Kimura K 2000 Discretization of the Euler top *J. Phys. Soc. Japan* **69** 627–30

[15] Hirota R, Kimura K and Yahagi H 2001 How to find the conserved quantities of nonlinear discrete equations *J. Phys. A: Math. Gen.* **34** 10377–86

[16] Iatrou A and Roberts J A G 2002 Integrable mappings of the plane preserving biquadratic invariants curves II *Nonlinearity* **15** pp 459–89

[17] Jacobi C G J 1828 Über die Anwendung der elliptischen Transcendenten auf ein bekanntes problem der elementargeometrie *J. Reine Angew. Math.* **3** 376–89

[18] John F 1941 The Dirichlet problem for a hyperbolic equation *Am. J. Math.* **63** 141–54

[19] Maxwell J C 1854 Solutions of the problems *Cambridge Dublin Math. J.* 8 188–95

Maxwell J C 2003 *The Scientific Papers of James Clerk Maxwell* vol 1 (Mineola, NY: Dover) pp 74–80

[20] Lion G 1994 Variational aspects of Poncelet's theorem *Geometriae Dedicata* **52** 105–18
[21] Minesaki Y and Nakamura Y 1998 On an integrable discretization of integrable systems with a separatrix *Phys. Lett.* A **250** 300–10
[22] Minesaki Y and Nakamura Y 2002 A new discretization of the Kepler motion which conserves the Runge-lenz vector *Phys. Lett.* A **306** 127–33
[23] Nakazato H 2003 Remark on Poncelet's closure theorem *Bull. Fac. Sci. Technol. Hirosaki Univ.* **5** 1–10
[24] Petrera M and Suris Yu 2007 On the Hamiltonian structure of Hirota–Kimura discretization of the Euler top arXiv:0707.4382 [math-ph]
[25] Sachkov Yu L 2008 Maxwell strata in the Euler elastic problem *J. Dyn. Control Syst.* **14** 169–234
[26] Schoenberg I J 1980 On Jacobi–Bertrand's proof of a theorem of Poncelet *Studies in Pure Mathematics to the Memory of Paul Turan* (Budapest: Hungarian Academy of Sciences) pp 623–7
[27] Sogo K 2006 Variational discretization of Euler's elastica problem *J. Phys. Soc. Japan.* **75** 064007
[28] Spiridonov V and Zhedanov A 2007 Elliptic grids, rational functions, and Padé interpolation *Ramanujan J.* **13** 285–310
[29] Suris Yu B 2003 The problem of integrable discretization: Hamiltonian approach *Progress in Mathematics* vol 219 (Basel: Birkhuser)
[30] Tabachnikov S 2005 Geometry and billiards *Student Mathematical Library* vol 30 (Providence, RI: American Mathematical Society)
[31] Toda M 1989 *Theory of Nonlinear Lattices (Springer Series in Solid-State Sciences vol 20)* (Berlin: Springer)
[32] Veselov A P 1988 Integrable systems with discrete time and difference operators *Funct. Anal. Appl.* **22** 1–13 (Russian)
[33] Veselov A P 1991 What Is an Integrable Mapping? *What Is Integrability?* ed V E Zakharov (Berlin: Springer) pp 251–72
[34] Whittacker E T and Watson G N 1927 *A Course of Modern Analysis* 4th edn (Cambridge: Cambridge University Press)